Observables and Gauge Fixing in Spontaneously Broken Gauge Theories

Martin Lavelle[1] and David McMullan[2]

[1]Grup de Física Teòrica and IFAE
Edificio Cn
Universitat Autónoma de Barcelona
E-08193 Bellaterra (Barcelona)
Spain
email: lavelle@ifae.es

[2]School of Mathematics and Statistics
University of Plymouth
Drake Circus, Plymouth, Devon PL4 8AA
U.K.
email: d.mcmullan@plymouth.ac.uk

Abstract   Gauge fixing and the observable fields for both abelian and non-abelian gauge theories with spontaneous breaking of gauge symmetry are studied. We explicitly show that it is possible to globally fix the gauge in the broken sector and hence construct physical fields even in the non-abelian theory. We predict that any high temperature restoration of gauge symmetry will be accompanied by a confining transition.

In a recent series of papers[1−3] we have investigated the physical degrees of freedom in abelian and non-abelian gauge theories and the intimately related question of gauge fixing. For Quantum Electrodynamics (QED) the physical degrees of freedom are the two transverse photon polarisations and the observed electron. This electron is not the Lagrangian fermion, which is neither gauge invariant nor associated with an electric field. In fact the physical electron is this fermion accompanied by a non-local photonic cloud[4]. Use of these physical degrees of freedom yields a description of QED that is both gauge invariant and infrared finite already at the level of the Green's functions.

For Quantum Chromodynamics (QCD), or indeed any non-abelian gauge theory, we have proven[3] that it is impossible to globally construct observables describing the fundamental fields (see also Ref. 5). The obstruction is the Gribov ambiguity[6]. A direct consequence of this is that it is impossible to observe quarks and gluons. Local expressions for physical quarks and gluons can, however, be developed[3]. The scale at which these expressions display a gauge dependence is the confinement scale. The calculation of this scale offers a new approach to determining the sizes of hadrons.

A question which naturally arises, and which we will address in this paper, is how can we reconcile the above obstruction to the observability of the fundamental fields in any non-abelian gauge theory with the existence of leptons, and the W and Z bosons? Our resolution of this apparent difficulty will be to show that in a spontaneously broken gauge theory one can use the Higgs matter field to fix the gauge and so circumvent the Gribov ambiguity. We stress that it is not sufficient merely to introduce scalar matter to achieve this result, additionally *one must be in the spontaneously broken sector of the theory*. This has consequences for the high temperature regime.

In QED the physical fields are

$$A_i^{\text{phys}} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) A^j \,, \tag{1}$$

and

$$\psi_{\text{phys}} = \exp\left(ig \frac{\partial_i A_i}{\nabla^2}(x) \right) \psi(x) \,, \qquad \bar{\psi}_{\text{phys}} = \exp\left(-ig \frac{\partial_i A_i}{\nabla^2}(x) \right) \bar{\psi}(x) \,. \tag{2}$$

These are straightforwardly seen to be gauge invariant. In scalar electrodynamics the Lagrangian is

$$\mathcal{L} = (\partial_\mu + ig A_\mu)\phi \, (\partial^\mu - ig A^\mu)\phi^* - m^2 \phi^* \phi - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \tag{3}$$

This is invariant under the gauge transformations

$$A_\mu \to A_\mu^U := A_\mu + \frac{1}{ig} U^{-1} \partial_\mu U \,, \qquad \phi \to \phi^U := U^{-1} \phi \,, \tag{4}$$

where $U(x) = e^{-ig\Lambda(x)}$. The physical, gauge invariant scalar fields are

$$\phi_{\mathrm{phys}}(x) = \exp\left(ig \frac{\partial_i A_i}{\nabla^2}(x)\right) \phi(x) \,, \qquad \phi_{\mathrm{phys}}^*(x) = \exp\left(-ig \frac{\partial_i A_i}{\nabla^2}(x)\right) \phi^*(x) \,. \tag{5}$$

They generate the electric field associated with a static charge. Note that the physical photons are still given by (1) in the scalar theory. These physical fields may be obtained from the Lagrangian fields by a gauge transformation

$$A_i^{\mathrm{phys}}(x) = A_i^h(x) \,, \qquad \phi_{\mathrm{phys}}(x) = \phi^h(x) \,, \qquad \phi_{\mathrm{phys}}^*(x) = \phi^{*\,h}(x) \,, \tag{6}$$

where h is a field dependent element of the gauge group that must itself behave as

$$h(x) \to h^U(x) = h(x) U^{-1}(x) \,, \tag{7}$$

under an arbitrary gauge transformation. It is readily shown that these physical fields are then gauge invariant[3]. The exact form of h that yields the above physical fields may be read off from (5) and (6).

In the abelian Higgs model one adds a potential term $\lambda(\phi^*\phi)^2$ to the Lagrangian (3) and obtains

$$\begin{aligned}
\mathcal{L} = (\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 + 2g A_\mu \phi_1 \partial^\mu \phi_2 - 2g A_\mu \phi_2 \partial^\mu \phi_1 + g^2 A_\mu A^\mu (\phi_1^2 + \phi_2^2) \\
- \lambda(\phi_1^2 + \phi_2^2)^2 - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,,
\end{aligned} \tag{8}$$

where we have also made a change of variables, $\phi = \phi_1 + i\phi_2$. To obtain spontaneous symmetry breaking[8] one assumes that m^2 is negative. It follows that the Higgs potential now develops a minimum at

$$|\phi| = a = \sqrt{\frac{-m^2}{2\lambda}} \,. \tag{9}$$

To expand around the minimum we write $\phi = \varphi + a$. In terms of these new variables (with the substitution $\varphi = \varphi_1 + i\varphi_2$) the Lagrangian reads

$$\begin{aligned}
\mathcal{L} = (\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2 + 2g A_\mu \varphi_1 \partial^\mu \varphi_2 - 2g A_\mu \varphi_2 \partial^\mu \varphi_1 + g^2 A_\mu A^\mu (\varphi_1^2 + \varphi_2^2) \\
+ 2ga A_\mu \partial^\mu \varphi_2 + 2g^2 a A_\mu A^\mu \varphi_1 + g^2 a^2 A_\mu A^\mu - \lambda(\varphi_1^2 + \varphi_2^2)^2 \\
- 4a\lambda(\varphi_1^2 + \varphi_2^2)\varphi_1 - 4a^2 \lambda \varphi_1^2 - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,.
\end{aligned} \tag{10}$$

In terms of these shifted fields the gauge transformation (4) becomes $\varphi \to e^{ig\Lambda}(\varphi + a) - a$, which yields

$$
\begin{aligned}
\varphi_1 &\to (\varphi_1 + a)\cos(g\Lambda) - \varphi_2 \sin(g\Lambda) - a\,, \\
\varphi_2 &\to (\varphi_1 + a)\sin(g\Lambda) + \varphi_2 \cos(g\Lambda)\,.
\end{aligned}
\tag{11}
$$

Hence the physical, gauge invariant fields with zero vacuum expectation value are

$$
\begin{aligned}
\varphi_{1\,\mathrm{phys}} &= (\varphi_1 + a)\cos\left(g\frac{\partial_i A_i}{\nabla^2}\right) - \varphi_2 \sin\left(g\frac{\partial_i A_i}{\nabla^2}\right) - a\,, \\
\varphi_{2\,\mathrm{phys}} &= (\varphi_1 + a)\sin\left(g\frac{\partial_i A_i}{\nabla^2}\right) + \varphi_2 \cos\left(g\frac{\partial_i A_i}{\nabla^2}\right)\,.
\end{aligned}
\tag{12}
$$

Equations (1) and (12) provide a parameterisation of the physical fields in the abelian Higgs model valid in both the broken ($a \neq 0$) and unbroken ($a = 0$) sectors. Although these fields make clear the physical content of the theory in the unbroken sector, they do not, however, provide the most natural parameterisation of the broken sector, as these expressions reduce in the Coulomb gauge to the transverse photon (1) and the pair of scalar fields (φ_1, φ_2), thus the expected content of the spontaneously broken theory, i.e., a massive gauge boson and a single scalar field, is not immediately apparent. In the broken sector an alternative basis is, however, also available since we can now construct the field dependent group element h from the scalar matter fields. Choosing

$$
h = \exp\left(-i\tan^{-1}\left(\frac{\varphi_2}{\varphi_1 + a}\right)\right)\,,
\tag{13}
$$

which is straightforwardly demonstrated to fulfill (7), we find that

$$
\begin{aligned}
A_i^{\mathrm{phys}} &= A_i + \frac{1}{g}\partial_i \tan^{-1}\left(\frac{\varphi_2}{\varphi_1 + a}\right)\,, \\
\varphi_{1\,\mathrm{phys}} &= \left((\varphi_1 + a)^2 + \varphi_2^2\right)^{\frac{1}{2}} - a\,,
\end{aligned}
\tag{14}
$$

and

$$
\varphi_{2\,\mathrm{phys}} = -\sin\left(\tan^{-1}\left(\frac{\varphi_2}{\varphi_1 + a}\right)\right)(\varphi_1 + a) + \cos\left(\tan^{-1}\left(\frac{\varphi_2}{\varphi_1 + a}\right)\right)\varphi_2 \equiv 0\,,
\tag{15}
$$

where the last relationship is a trigonometric identity. With respect to these fields the physical content of the broken sector of the theory becomes transparent. In particular working in the unitary gauge, where we set φ_2 to zero, we see that these fields just reduce to the three physical photon components and one scalar, physical Higgs field. From the

4

gauge invariance of the theory and of the physical fields we know that this interpretation of the theory must hold in any gauge, although it may be, as we saw in Coulomb gauge, obscured.

Our obtaining this interpretation of the abelian Higgs model relied upon our ability to use the unitary gauge and to make the gauge transformation (13), which we now want to investigate. To this end we now recall that a gauge fixing condition provides a means for picking out a representative from each gauge orbit in the configuration space. Clearly a good gauge fixing term should only pick out one such representative from each orbit. In both the abelian and non-abelian Higgs model the configuration space of the Yang-Mills and scalar fields is topologically trivial, hence a gauge fixing condition will be good if it does not "turn back" on itself as it slices through the orbits. The Faddeev-Popov functional determinant provides a measure of this: for a good choice of gauge it does not vanish for any configuration. In a Hamiltonian approach[7] to the abelian theory this means that the Poisson bracket of the gauge condition and Gauss' law must not vanish for any allowed field configuration. If it vanishes for some configuration of fields we have a Gribov ambiguity[6]. It has been shown by Singer[6] that in non-abelian gauge theories, with some mild assumptions on the boundary conditions, all gauge fixings in the vector boson sector suffer from such a Gribov ambiguity. Indeed our account of confinement in QCD is based on the fact that if physical quarks could be defined then they could be used to construct a good gauge fixing — in contradiction to Singer's result. Since the unitary gauge offers us insight into the physical fields in a spontaneously broken gauge theory, it is natural to now ask if it is in fact an allowed gauge or if it suffers from a Gribov type problem.

From the Lagrangian (10) we obtain the Gauss law constraint

$$G(x) = -\partial_i \pi^i + g\pi_{\varphi_1}\varphi_2 - g\pi_{\varphi_2}\varphi_1 - ag\pi_{\varphi_2} \,, \tag{16}$$

where the $\pi_i, \pi_{\varphi_1}, \pi_{\varphi_2}$ are the conjugate momenta to the $A_i, \varphi_1, \varphi_2$ fields respectively. In both the broken and unbroken sectors of this abelian theory the Coulomb gauge, $\partial_i A_i = 0$, is an example of a good gauge condition since $\{G(x), \partial_i A_i(y)\} = \nabla^2 \delta(x-y)$ independently of a. This restates the lack of a Gribov problem with gauge fixing in this abelian theory.

For the unitary gauge we have

$$\{G(x), \varphi_2(y)\} = g(\varphi_1(x) + a)\delta(x-y) \,. \tag{17}$$

Now we recall that from finite energy considerations[8] we must have that the scalar fields tend to zero at infinity. Hence $\varphi_1 \equiv 0$ is an allowed configuration, but $\varphi_1 + a \equiv 0$ for

non-zero a is not. This means that the unitary gauge, $\varphi_2 = 0$, is acceptable in the broken sector, but not in the unbroken one. We will call this a Gribov problem[1] with the unitary gauge in the unbroken theory.

The existence of a Gribov ambiguity for the unitary gauge in the unbroken sector would seem to imply that this gauge can there have at best a perturbative validity. However, from the Lagrangian (10) we see that the photon propagator does not exist for vanishing a. It is thus clear that the unitary gauge in the unbroken sector of the abelian Higgs model is completely unacceptable. Similarly the field parameterisation (14) in the unbroken sector is not allowed because of the $\frac{1}{\varphi_1 + a}$ term which is singular for vanishing a, which shows that (13) is not a well-defined gauge transformation if the gauge symmetry is not broken.

The action of the gauge group on the Higgs fields, in this model, breaks the linear space into $U(1)$-orbits that are either circles or the exceptional point $\phi = 0$. Symmetry breaking implies that it is one of the non-trivial orbits that is the vacuum for the theory. Working in polar coordinates we write $\phi = (\varphi_1 + a) + i\varphi_2 = \rho \exp i\theta$, and the physical fields (14) are then

$$A_i^{\text{phys}} \to A_i + \frac{1}{g}\partial_i\theta \,,$$
$$\varphi_{1\,\text{phys}} \to \rho - a \,, \tag{18}$$

which simplifies even further in the unitary gauge, $\theta = 0$. The new coordinates, (ρ, θ), are ill defined at the origin $\rho = 0$. However, as we have seen, this point is only an allowed configuration for the unbroken theory. Hence the polar coordinates are a globally valid coordinate system in the broken sector of the theory, and this is what allows us to find the physical fields (18).

Before giving the details of how this abelian example can be extended to the non-abelian theory, it is useful to give a qualitative account of how we are to proceed. We know that in an abelian gauge theory we can fix the gauge and hence construct physical fields. In a non-abelian theory, the Higgs mechanism is used to reduce the symmetry; so if it can be reduced to an abelian group (as it is in the electro-weak theory) then we would expect to be able to construct physical fields.

More geometrically, in a non-abelian theory with structure group G, the Yang-Mills field is identified with the connections on a principal G-bundle P over the space time.

---

[1] More properly we should say that the unbroken theory has a Gribov problem with gauge fixing in the matter sector: for this abelian theory there is no Gribov problem if we gauge fix in the gauge sector.

As long as the structure group is non-abelian there will be a Gribov problem associated with gauge fixing, and hence an obstruction to constructing physical fields. The Higgs fields $\underline{\phi}$ takes values in some vector space V and can thus be viewed as cross sections of the associated bundle $P \times_G V$ — with fibres over the space-time now being the vector space V. The vector space V can be thought of as a collection of G-orbits, a typical example being when $G = \mathrm{SO}(n)$, and $V = \mathbb{R}^n$; in which case the orbits are the origin (with stability group $H = G$) and the $(n-1)$-spheres (with stability group $H = \mathrm{SO}(n-1)$). In symmetry breaking, the potential energy of the Higgs fields is such that the vacuum configuration corresponds to the Higgs fields being restricted to one of these orbits, which we identify with the coset space G/H, for some stability subgroup H of G. Choosing a point on an orbit is equivalent to gauge fixing in the matter sector. These vacuum solutions now correspond to cross sections of the associated bundle $P \times_G G/H$, which can be identified with the quotient P/H. Now cross sections of P/H correspond to reductions of the structure group from G to H (see, for example, the theorem on page 385 of Ref. 9). Thus, if we can use gauge fixing in the matter sector to reduce the structure group to an abelian group, then the residual gauge symmetry can be dealt with in the gauge sector and we will not encounter any Gribov problem. We shall now demonstrate through an explicit calculation that this is what happens in the Salam-Weinberg model.

We now consider the case of spontaneously broken SU(2) gauge theory. The scalar fields $\underline{\phi}$ are now two-component, complex column vectors. The orbits in the target space are three spheres which we can identify with SU(2), thus having a trivial stability group. (This is essentially the weak sector of the standard model with the Weinberg angle set to zero for simplicity.) The gauge transformations of the fields are

$$A_i \equiv A_i^a \tau^a \to A^U = U A_i U^{-1} - \frac{i}{g} U \partial_i U^{-1} \,,$$

$$\underline{\phi} \to \underline{\phi}^U = U \underline{\phi} \,. \tag{19}$$

The shifted scalar fields $\underline{\varphi} = \underline{\phi} - \underline{a}$ are taken to have zero expectation value, and we explicitly write

$$\underline{\varphi} = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix} \,, \tag{20}$$

where a is real. Under the gauge transformation (19), these shifted fields transform as

$$\underline{\varphi} \to \underline{\varphi}^U = U(\underline{\varphi} + \underline{a}) - \underline{a} \,. \tag{21}$$

7

The physical fields are then identified with

$$A_i^{\text{phys}} = A^h := hA_ih^{-1} - \frac{i}{g}h\partial_ih^{-1}$$

$$\underline{\varphi}_{\text{phys}} = \underline{\varphi}^h := h(\underline{\varphi} + \underline{a}) - \underline{a}, \tag{22}$$

where the field dependent group element h satisfies (7). A direct calculation shows that

$$h = \frac{1}{\sqrt{\varphi_1^2 + \varphi_2^2 + (\varphi_3 + a)^2 + \varphi_4^2}}\begin{pmatrix} \varphi_3 + a + i\varphi_4 & -\varphi_1 - i\varphi_2 \\ \varphi_1 - i\varphi_2 & \varphi_3 + a - i\varphi_4 \end{pmatrix}, \tag{23}$$

satisfies this requirement, and can hence be used to generate gauge invariant fields. From (22) and (23) we find that the physical gauge and scalar fields are

$$\begin{aligned}
\underline{\varphi}_{\text{phys}} &= \begin{pmatrix} 0 \\ (\varphi_1^2 + \varphi_2^2 + (\varphi_3 + a)^2 + \varphi_4^2)^{\frac{1}{2}} - a \end{pmatrix}, \\
A^1_{i\,\text{phys}} &= \frac{1}{\Phi^2}\left[-A_i^1(\varphi_1^2 - \varphi_2^2 - \varphi_3^2 + \varphi_4^2) + 2A_i^2(\varphi_1\varphi_2 + \varphi_3\varphi_4) + 2A_i^3(\varphi_1\varphi_3 - \varphi_2\varphi_4)\right] \\
&\quad + \frac{1}{g\Phi}\left[\varphi_4\partial_i\varphi_1 - \varphi_1\partial_i\varphi_4 + \varphi_3\partial_i\varphi_2 - \varphi_2\partial_i\varphi_3\right], \\
A^2_{i\,\text{phys}} &= \frac{1}{\Phi^2}\left[A_i^2(\varphi_1^2 - \varphi_2^2 + \varphi_3^2 - \varphi_4^2) + 2A_i^1(\varphi_1\varphi_2 - \varphi_3\varphi_4) - 2A_i^3(\varphi_2\varphi_3 + \varphi_1\varphi_4)\right] \\
&\quad + \frac{1}{g\Phi}\left[\varphi_3\partial_i\varphi_1 - \varphi_1\partial_i\varphi_3 + \varphi_2\partial_i\varphi_4 - \varphi_4\partial_i\varphi_2\right], \\
A^3_{i\,\text{phys}} &= \frac{1}{\Phi^2}\left[-A_i^3(\varphi_1^2 + \varphi_2^2 - \varphi_3^2 - \varphi_4^2) - 2A_i^1(\varphi_1\varphi_3 + \varphi_2\varphi_4) + 2A_i^2(\varphi_2\varphi_3 - \varphi_1\varphi_4)\right] \\
&\quad + \frac{1}{g\Phi}\left[\varphi_4\partial_i\varphi_3 - \varphi_3\partial_i\varphi_4 + \varphi_2\partial_i\varphi_1 - \varphi_1\partial_i\varphi_2\right].
\end{aligned} \tag{24}$$

where $\Phi^2 = \varphi_1^2 + \varphi_2^2 + (\varphi_3 + a)^2 + \varphi_4^2$. The physical scalar fields are those of the unitary gauge. It is easily seen that, in the broken sector, this is indeed a good gauge. Recalling that the Faddeev-Popov matrix is the Poisson bracket of Gauss' law, the generator of infinitesimal gauge transformations, with the gauge function, it is easily seen to be invertible. For example, we have

$$\{G^a\epsilon^a, \varphi^1\} \sim g\left(\epsilon_3\varphi_2 + \epsilon_1\varphi_4 - \epsilon_2(\varphi_3 + a)\right) \tag{25}$$

and it is clear that the unitary gauge is a 'good' gauge even in this nonabelian model. In unitary gauge the physical bosons reduce to $\underline{\varphi}_{\text{phys}} = \begin{pmatrix} 0 \\ \varphi_3 \end{pmatrix}$ and $A^a_{i\,\text{phys}} = A_i^a$, as one would expect.

Our construction of these physical, gauge-invariant fields in a non-abelian theory depended crucially upon our ability to fix the unitary gauge. Since we expect gauge symmetry to be restored at some finite temperature, it will then no longer be possible to fix a gauge in the scalar sector at all. In the vector boson sector of a non-abelian theory only a perturbative gauge fixing is possible[3] and so only perturbatively physical fields can be constructed in the unbroken sector.

In the full Salam-Weinberg model the fact that we are dealing with a non-simple group means that we are left with an unbroken $U(1)$ symmetry where gauge fixing is possible and a complete gauge fixing may be constructed by a trivial extension of the above analysis: partially in the abelian gauge boson sector and the rest, as above, in the scalar, Higgs sector. Thus we may circumvent the Gribov ambiguity in the non-abelian part of the Salam-Weinberg model and build up physical, massive W, Z and Higgs bosons in the manner of (24). We also predict that at high temperatures in the unbroken sector of the weak interaction not only the non-abelian vector bosons and scalars but also all weakly interacting fundamental particles, such as the electron, will be confined.

In conclusion we have seen that, when coupling to scalar fields, one can fix the gauge in the scalar matter sector if the gauge symmetry is spontaneously broken. Since this holds even for non-abelian gauge theories, it offers a mechanism for avoiding the Gribov ambiguity in the broken sector of the standard model. This ability to fix the gauge allows us to construct physical, gauge invariant fields. In particular is is possible to describe the W and Z bosons so long as the gauge symmetry is broken. This is to be contrasted with the impossibility of constructing physical quarks and gluons in QCD. Our ability to construct physical fields to describe electrons, photons and the like in QED and their equivalents in the broken sector of the standard model together with the lack of physical quarks and gluons is in satisfyingly complete agreement with experiment. At high temperature when the gauge symmetry is restored we predict that the W and Z bosons and all other fundamental weakly interacting fields will not be observables. This could have consequences for studies of the early universe.

Acknowledgements

References

[1] M. Lavelle and D. McMullan, *Phys. Rev. Lett.* **71** (1993) 3758.

[2] M. Lavelle and D. McMullan, *Phys. Lett.* **312B** (1993) 211.

[3] M. Lavelle and D. McMullan, *Phys. Lett.* **329B** (1994) 68.

[4] P.A.M. Dirac, "Principles of Quantum Mechanics", (OUP, Oxford, 1958), page 302.

[5] S.V. Shabanov, *Mod. Phys. Lett.* **A6** (1991) 909; *Phys. Lett.* **255B** (1991) 398;
     L.V. Prokhorov and S.V. Shabanov, *Int. J. Mod. Phys.* **A7** (1992) 7815.

[6] V.N. Gribov, *Nucl. Phys.* **B139** (1978) 1;
     I. Singer, *Commun. Math. Phys.* **60** (1978) 7.

[7] see, for example, R. Jackiw, in "Current Algebra and Anomalies",
     by S.B. Treiman et al. (World Scientific, Singapore, 1987).

[8] see, for example, L.H. Ryder, "Quantum Field Theory", (CUP, Cambridge, 1985).

[9] Y. Choquet-Bruhat, C. Dewitt-Morette and M. Dillard-Bleick, "Analysis, Manifolds and Physics,
     Revised Edition", (North Holland, Amsterdam, 1982).